10.6 Models Cont.

Savings account earns 6% interest per year compounded continuously. In addition continuous withdrawals are made from the account at the rate of \$900 per year.

To model this:

$$\frac{d\,\text{Stuff}}{dt} = \begin{bmatrix}\text{Stuff in} \\ \text{rate}\end{bmatrix} - \begin{bmatrix}\text{stuff} \\ \text{out rate}\end{bmatrix}$$

Stuff in: Account earns 6% interest compounded continuously so

$$\begin{matrix}\text{Rate of} \\ \$ \text{ in}\end{matrix} = 0.06y$$

Stuff out: we remove \$900 per year so

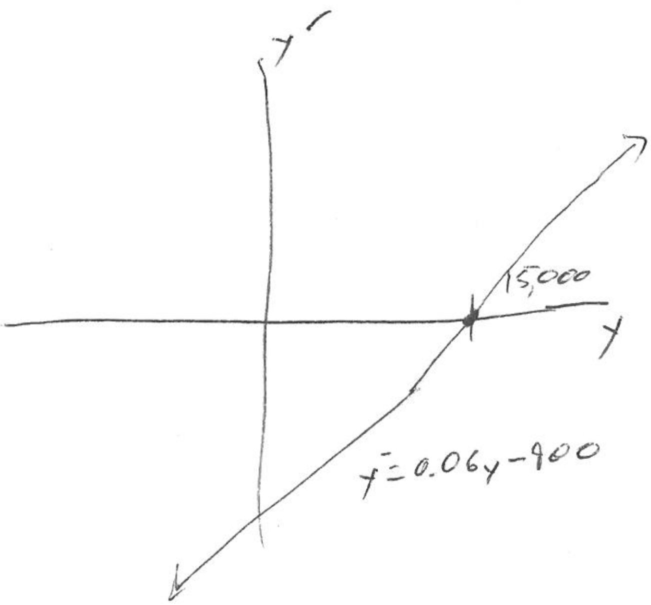
$$\begin{matrix}\text{Rate} \\ \text{out}\end{matrix} = 900$$

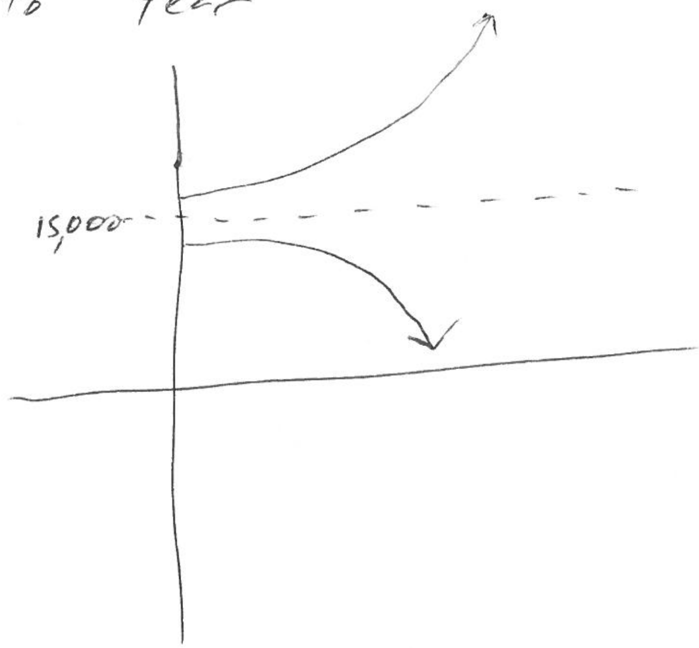
$$\frac{dy}{dt} = 0.06y - 900$$

Constant solution at $0.06y - 900 = 0$

$$y = \frac{900}{.06} = 15,000$$

If initial amount in account is \$15,000 it will stay \$15,000 year to year





If initial amount > \$15,000 the account will grow without bound

If initial amount < \$15,000 the account will decrease (realistically stop at \$0)

Mixing Saltwater

- We have a flask containing 3 L of salt water. Supposing water containing 25 g of salt per liter is pumped in at a rate of 2 Liters per hour. The mixture is stirred and pumped out at the same rate.

What differential equation models this process?

$$\frac{d\ \text{Salt}}{dt} = \left[\begin{matrix}\text{Salt in}\\ \text{rate}\end{matrix}\right] - \left[\begin{matrix}\text{Salt out}\\ \text{rate}\end{matrix}\right]$$

Salt in: $\frac{\text{salt}}{\text{hour}}$ is what we want

know $\frac{25\text{g salt}}{\text{Liter}}$ pumped in at $\frac{2\text{ Liters}}{\text{hour}}$

$$\frac{25\text{g Salt}}{\cancel{\text{Liter}}} \cdot \frac{2\ \cancel{\text{Liters}}}{\text{hour}} = \boxed{\frac{50\text{g Salt}}{\text{hour}}} \quad \text{In}$$

Salt out: y = grams salt in solution

concentration y g Salt $\div$ 3 Liters in flask

$= \frac{y \text{ g Salt}}{3 \text{ Liters}}$ pumped out at $\frac{2 \text{ Liters}}{\text{hour}}$

$$\frac{y \text{ g Salt}}{3\ \cancel{\text{Liters}}} \cdot \frac{2\ \cancel{\text{Liters}}}{\text{hour}} = \frac{2y}{3}\ \frac{\text{g Salt}}{\text{hour}}$$

$$\boxed{y' = 50 - \frac{2}{3}y}$$

Steady state at 75 g of salt
How does it behave?

In a tropical forest leaf litter accumulates at a rate of 10 grams ~~[illegible]~~ per cm^2 per year and decomposes at 80% per year. What differential equation models this ~~grams of leaf litter~~ ~~[illegible]~~ process where $f(t)$ is grams of leaf litter per cm^2?

Leaf litter in : 10

Leaf litter out : $0.8\, f(t)$

$$\frac{df}{dt} = 10 - 0.8\, f(t)$$

what is the ~~[illegible]~~ constant solution and how ~~[illegible]~~ ~~[illegible]~~ do the solutions behave qualitatively?

$0 = 10 - 0.8y$

$y = \frac{10}{.8} = 12.5$ grams/cm^2

Fish with harvesting.

The fish population in a pond with carrying capacity 1000 is modeled by

$$\frac{dN}{dt} = \frac{.4}{1000}(1000-N)$$

The owner wants to allow people to come catch fish and initially decides to allow 75 fish total to be caught per year.

1.) If he starts this program when his fish population is at 275 is this sustainable? ~~or will the population of the~~ Will the fish population come close ~~the~~ to the carrying capacity of the pond?

2.) What is the smallest number of fish they could have in the pond and still allow 75 to be caught each year and not run out of fish?

3.) What is the maximum # of fish they could allow to be caught each year and not deplete the lake? When should they start this program?

$$\frac{d\,\text{Fish}}{dt} = \text{Fish In} - \text{Fish Out}$$

$$\frac{dN}{dt} = \frac{.4N}{1000}(1000-N) - 75$$

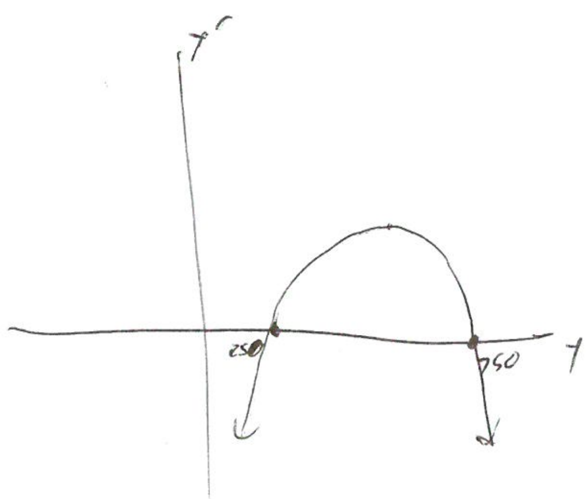
$$\frac{dN}{dt} = .4N - \frac{.4}{1000}N^2 - 75$$

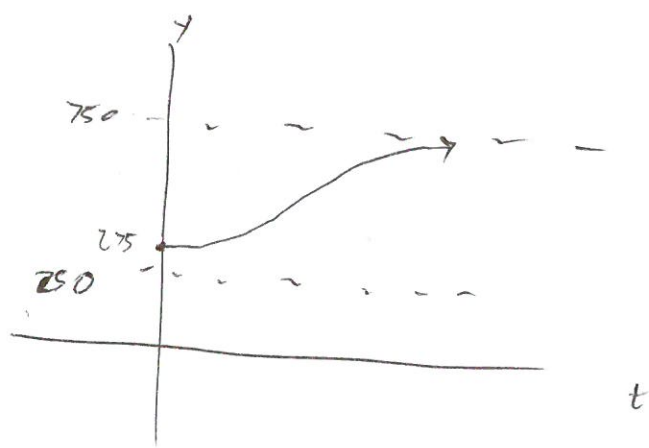
has roots when $0 = \frac{-.4}{1000}N^2 + .4N - 75$

Quadratic Eq: $\dfrac{-.4 \pm \sqrt{.4^2 - 4\left(\frac{-.4}{1000}(-75)\right)}}{2\cdot\left(\frac{-.4}{1000}\right)}$

$$= 500 \pm \frac{\sqrt{.04}}{2\left(\frac{-.4}{1000}\right)} = 500 \pm 250$$

Constant solution at $N = 250$
$N = 750$



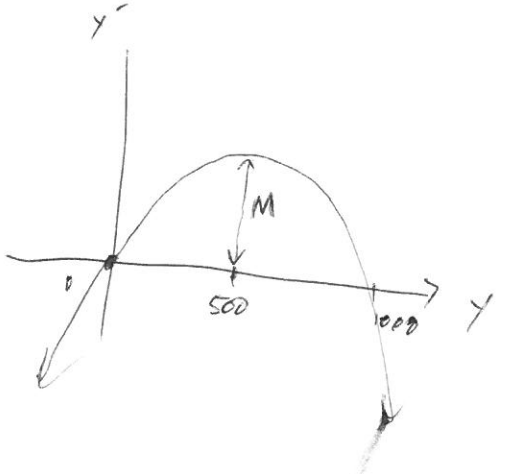


① Is sustainable, Fish population will near 750

② Smallest # of fish to start program would be 250, but does not leave much wiggle room for over fishing.

To find max # of fish that could be removed

$$\frac{dN}{dt} = \frac{.4N}{1000}(1000 - N)$$



removing fish causes y-y' plot to move downward.

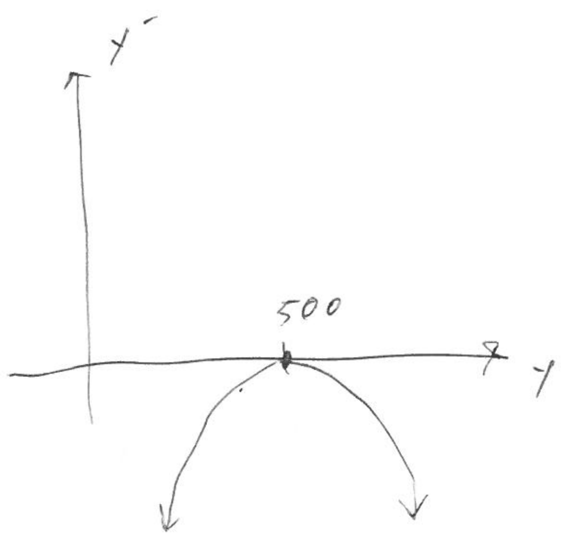
Need to make max growth a constant solution: Subtract M

Happens at $N = 500$

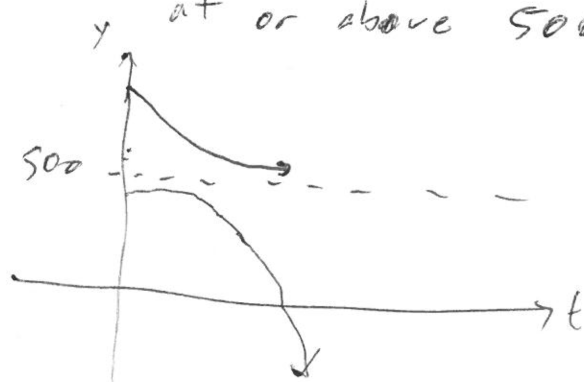
$\frac{.4N}{1000}(1000-N)$ when $N=500$

$$\frac{.4(500)}{1000}(1000-500) = \frac{.4}{2}(500) = 100$$

$$\frac{dN}{dt} = \frac{.4}{1000}N(1000-N) - 100$$



Can harvest 100 per year once fish population is at or above 500 fish



Savings with Deposits:

Savings account with 5% interest yearly
and deposits ~~$10,000~~ $1000 per year

Set up a differential equation describing the growth of money

$$\frac{dy}{dt} = .05y + 1000$$

Solve the differential equation and assume $f(0) = 0$.
How much will be in the account after 10 years?

$$\int \frac{dy}{.05y + 1000} = \int dt$$

$$\frac{1}{.05} \ln|0.05y + 1000| = t + C$$

$$\ln|0.05y + 1000| = .05t + C$$

$$0.05y + 1000 = Ce^{.05t}$$

$$y = \frac{1}{0.05}\left(Ce^{.05t} - 1000\right)$$

$$y = Ce^{.05t} - 20{,}000$$

$$y(0) = C - 20{,}000 = 0 \quad \Rightarrow \quad C = 20{,}000$$

$$y = 20000\left(e^{.05t} - 1\right) \qquad y(10) = 20000\left(e^{.5} - 1\right) \approx \$12{,}974$$