10.2 Separation of Variables

How to solve a class of differential equations of the form

$$y' = p(t)q(y)$$

Called "seperable" because we can seperate the variables

$$y' = \frac{3t^2}{y^2} \Rightarrow \boxed{y^2 y' = 3t^2}$$

y's on one side, t's on the other.

ex

1. $y' = 5y$ seperable $\Rightarrow$ $\frac{y'}{\cancel{y}} = 5$ $p(t) = 5$

2. $y' = 3t + 4$ seperable $\Rightarrow$ $y' = 3t + 4$ $q(y) = 1$

3. $y' = e^{-y}(2t+1)$ seperable $\Rightarrow$ $e^{y} y' = 2t + 1$

4. $y' = 5y - 2t$ not seperable
   cannot write $5y - 2t$ as $p(t)q(y)$

How to find solutions:

$$y' = \frac{3t^2}{y^2} \Rightarrow y^2 y' = 3t^2$$

write y' as $\frac{dy}{dt}$

$$y^2 \frac{dy}{dt} = 3t^2$$

$$\int y^2 \frac{dy}{\cancel{dt}} \cancel{dt} = \int 3t^2 \, dt$$

not exactly canceling, but we can think of it that way for simplicity.

$$\int y^2 \, dy = \int 3t^2 \, dt$$

$$\frac{1}{3} y^3 + C_1 = t^3 + C_2$$

$$\frac{1}{3} y^3 = t^3 + (C_2 - C_1)$$

$$\frac{1}{3} y^3 = t^3 + C_3$$

$C_2 - C_1$ is just a constant, so only need $+C$ on one side

$$y^3 = 3t^3 + 3C_3$$

since $3C_3$ is still a constant, we can just rewrite it as C

$$y^3 = 3t^3 + C$$

$$\boxed{y = \sqrt[3]{3t^3 + C}}$$

ex

$$y' = t^3 y^2 + y^2$$

$$y' = (t^2+1) y^2$$ seperable

$$\frac{1}{y^2}\frac{dy}{dt} = t^3 + 1$$

$$\int \frac{1}{y^2}\, dy = \int t^3 + 1 \, dt$$

$$-y^{-1} = \frac{t^4}{4} + t + C$$

$$\boxed{y = \frac{-1}{\frac{1}{4}t^4 + t + C}}$$

does this give all the solutions?

no!

consider $y = 0$

$\Rightarrow y' = 0$ check:

$$0 = t^3 \cdot 0 + 0 \quad \checkmark$$

ex

Solve the IVP:

$$y' = e^{-y}(2t+1) \qquad y(0) = 1$$

$$e^{y} y' = 2t+1$$

$$\int e^{y}\,dy = \int 2t+1\,dt$$

$$e^{y} = t^2 + t + C$$

$$y = \ln(t^2+t+C)$$

using $y(0) = 1$

$$\cancel{1 = \ln(t^2+t)} \qquad 1 = \ln(C)$$

$$\Rightarrow e = C$$

$$\boxed{y = \ln(t^2 + t + e)}$$

ex

$$y' = ky$$

$$\int \frac{dy}{y} = \int k\, dt$$

$$\ln(y) = kt + C$$

$$y = e^{kt+C}$$

$$y = Ce^{kt}$$

ex

$$y' = k(12-y)$$

$$\frac{y'}{12-y} = k$$

$$\int \frac{1}{12-y}\, dy = \int k\, dt$$

$$\ln|12-y| = kt + C$$

$$12 - y = Ce^{kt}$$

$$y = 12 - Ce^{kt}$$

ex/ $y' = \frac{\ln t}{t y}$

ex/ $\frac{dy}{dt} = \frac{te^t}{y}$

use integration by parts

ex/ $y' = (y-3)^2 \ln t$

This is also integration by parts.
Hint: u = ln(x), dv = 1

ex/ $\frac{dy}{dt} = \frac{t^2 y^2}{t^3+8}$

ex/ $\frac{dy}{dx} = \frac{\ln x}{\sqrt{x}\, y}$, $y(1) = 4$

use integration by parts

ex/ $y' = y^2 - e^{3t} y^2$, $y(0) = 1$