5.1 Exponential Growth & Decay.

Growth of Rabbit population:

The more Rabbits there are, the faster the population grows.

Model by claiming that the growth rate is proportional to the population size.

proportional: Let k be a constant.

Ideal Gas Law:

$P = \frac{kT}{V}$

- y and x are directly proportional if $y = kx$

  as $x \uparrow$ $y \uparrow$

  as $x \downarrow$ $y \downarrow$

  Pressure & Temperature

- y and x are inversely proportional if $y = \frac{k}{x}$

  as $x \uparrow$ $y \downarrow$

  as $x \downarrow$ $y \uparrow$

  Pressure and Volume

Growth rate: Derivative of population with respect to time.

rate of change

$y'(t)$

Population Size: $y(t)$

If growth rate of population is proportional to the current population the for some constant k

$$y'(t) = k\,y(t) \quad \text{or} \quad y' = ky$$

Q: what is y?

"A function whose derivative is itself times k"

Claim $y = e^{kt}$ fits this description

$$y' = k\,e^{kt} = k\,y$$

Also:

$$y = P_0 e^{kt}$$

$$y' = k\left(P_0 e^{kt}\right) = ky$$

Review

$$\ln(e^t) = t$$

$$e^{\ln(t)} = t$$

$$e^{ab} = e^a e^b$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(a^b) = b\ln(a)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

ex/ ~~ex~~ A bacteria culture grows at a rate proportional to its size. We start with 100 and 5 hours later have 500 bacteria.

1. Find k "the growth constant
2. When does the population double in size?
3. what is the growth rate at $t=5$
4. what is the growth rate ~~after~~ when 1,000 bacteria are present.

1. $y = P_0 e^{kt}$ we know $t=0$ $y=100$ and $t=5$ $y=500$

$$y(0) = P_0 e^0$$

$$\boxed{100 = P_0}$$

$$\Rightarrow y = 100e^{kt}$$

$$y(5) = 100e^{5k} = 500$$

$$\Rightarrow 5 = e^{5k}$$

$$\ln 5 = 5k$$

$$\boxed{k = \frac{\ln 5}{5} \approx 0.322}$$

2. Doubling
$y = 100 e^{0.322t}$

want t_1 s.t $y(t_1) = 200$

$$200 = 100 e^{0.322t}$$

$$2 = e^{0.322t}$$

$$\ln(2) = 0.322t$$

$$\frac{\ln(2)}{0.322} = t \approx 3.41 \text{ hours.}$$

3. Growth rate when $t = 5$

$$y'(5)$$

$$y' = 100(0.322)e^{0.322t}$$

$$y'(5) = 100(0.322)e^{0.322(5)} \approx 161 \ \frac{\text{bacteria}}{\text{hour}}$$

4. Growth rate when 1,000 bacteria.

Recall $y' = ky$

$$y' = (0.322)y$$

$$y' = (0.322)(1000) = 322 \ \frac{\text{bacteria}}{\text{hour}}$$

ex

Compound Intrest

\$9,000 is put into an account that earns 7% intrest per year continuously compounded.

(1) How much will we have after 20 years?

(2.) what is the growth rate when there is \$10,000 in the account?

(3). \$100,000 ?

(1.) ~~[illegible]~~

Compound intrest ~~[illegible]~~ follows $y' = ky$ here intrest = k

$$y' = 0.07y$$

$$y = P_0 e^{0.07t}$$

$$9000 = P_0 e^0$$

$$y = 9000 e^{0.07t}$$

$$y(20) = 9000 e^{0.07(20)} = \$36,496.80$$

(2.) $y' = 0.07(10{,}000) = \$700 / \text{year}$

(3.) $y' = 0.07(100{,}000) = \$7{,}000 / \text{year}$

Q/ How long to double if you start with \$10,000 in your account at 7% ?

How long if you start with \$10 ?

ex/ Exponential Decay

Happens when k is negative

The decay constant for a radioactive element is 0.0244

How long will it take to decay to half its mass?

$$y = P_0 e^{-0.0244t}$$

$$\tfrac{1}{2}P_0 = P_0 e^{-0.0244t}$$

$$\tfrac{1}{2} = e^{-0.0244t}$$

$$\ln(\tfrac{1}{2}) = -0.0244t$$

$$t = \frac{\ln(.5)}{-0.0244} \approx 28 \text{ years}.$$

"called the half-life"