Taylor Series Review and More Examples

The Taylor Series for $f(x)$ centered @ $x=0$ is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{for } |x|<1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

ex $\quad x(e^x - 1)$

$$e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1$$

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$x(e^x - 1) = x\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$x(e^x - 1) = x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots$$

ex

$$e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

For class:

Find Taylor series of

$$\frac{1}{1+x^3} \quad \text{and} \quad \frac{x^2}{1+x^3}$$

We can use Taylor series to find (or approximate) integrals of functions we normally can't integrate.

ex

$y = e^{-\frac{x^2}{2}}$ has no ~~formula~~ simple formula for its antiderivative.

Can use Taylor series to approximate $\int e^{-\frac{x^2}{2}}\,dx$

$$e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$e^{-\frac{x^2}{2}} = \left(1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2!} + \frac{\left(-\frac{x^2}{2}\right)^3}{3!} + \frac{\left(-\frac{x^2}{2}\right)^4}{4!} + \cdots\right.$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{2^2\cdot 2!} - \frac{x^6}{2^3\cdot 3!} + \frac{x^8}{2^4\cdot 4!} + \cdots$$

$$\int e^{-\frac{x^2}{2}}dx = \int\left(1 - \frac{x^2}{2} + \frac{x^4}{2^2\cdot 2!} - \frac{x^6}{2^3\cdot 3!} + \frac{x^8}{2^4\cdot 4!} + \cdots\right)dx$$

$$= x - \frac{x^3}{3\cdot 2} + \frac{x^5}{5\cdot 2^2\cdot 2!} - \frac{x^7}{7\cdot 2^3\cdot 3!} + \frac{x^9}{9\cdot 2^4\cdot 4!} + \cdots$$

$$+ C$$

ex

Approximate $\displaystyle\int_0^2 \frac{e^{-x}-1}{x}\,dx$

$$e^{x} - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} - 1 = -x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\frac{e^{-x}-1}{x} = -1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \cdots$$

$$\int_0^2 \frac{e^{-x}-1}{x}\,dx = \left(-x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \frac{x^4}{4\cdot 4!} - \cdots \right)\Big|_0^2$$

$$= -2 + \frac{2^2}{2\cdot 2!} - \frac{2^3}{3\cdot 3!} + \frac{2^4}{4\cdot 4!} - \cdots$$

ex T.S. for $x^2 e^{x^3}$ @ $x=0$

order matters:

$$e^x \xrightarrow{\times x^2} x^2 e^x \xrightarrow[\text{replace } x \text{ with } x^3]{} (x^3)^2 e^{x^3}$$

$$e^x \xrightarrow[\text{replace } x \text{ with } x^3]{} e^{x^3} \xrightarrow{\times x^2} x^2 e^{x^3}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots$$

$$x^2 e^{x^3} = x^2 + x^5 + \frac{x^8}{2!} + \frac{x^{11}}{3!} + \frac{x^{14}}{4!} + \cdots$$

Taylor Series for $\tan(x)$ @ $x=0$

is:

$$x + \frac{1}{3}x^2 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

what is the 5^{th} derivative of $\tan(x)$ at $x=0$?

$$f^{(5)}(0) = ?$$

look at x^5 term. $\rightarrow$ $\frac{2}{15}x^5 = \frac{f^{(5)}(0)}{5!}x^5$

so $$\frac{2}{15} = \frac{f^{(5)}(0)}{5!}$$

$$\Rightarrow \quad 5! \cdot \frac{2}{15} = f^{(5)}(0)$$

$$\frac{\cancel{5}\cdot 4\cdot \cancel{3}\cdot 2\cdot 1\cdot 2}{\cancel{15}} = 4\cdot 2\cdot 2 = \boxed{16}$$

ex

what is ~~the~~ $f^{(4)}(0)$ when $f(x) = e^{-\frac{x^2}{2}}$?

what about $f^{(5)}(0)$?

ex

Find T.S. expansion for $\int \sin(x^2)\,dx$

where

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$$\int \sin(x^2)\,dx = C + \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots$$