ex A patient receives 6 mg of a certain drug daily.

Each day the body eliminates 30% of the drug in its system

After extended treatment estimate the amount of drug in their system.

key: Assume infinite treatments to take infinite sum.

drug from a treatment in body after n days is ~~6(.3)~~ $6\cdot(.7)^n$

$$6 + 6(\cancel{.3}\,.7) + 6(\cancel{.3}\,.7)^2 + 6(\cancel{.3}\,.7)^3 + \cdots$$

$a = 6$

$r = .3$

$$\frac{6}{1-.3} = \frac{6}{.3} = 6\cdot\frac{10}{3} = \frac{60}{3}$$

$= 20$ mg

$0.4\overline{09} = 0.4 + 0.0\overline{09}$

$0.\overline{123}$

$$\frac{123}{1000} + \frac{123}{1000^2} + \frac{123}{1000^3}$$

$$\frac{\frac{123}{1000}}{1-\frac{1}{1000}} = \frac{123}{1000}\cdot\frac{1000}{999} = \frac{123}{999} = \frac{41}{333}$$

11.5 Taylor Series

Consider

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

Like a geometric series where $a = 1$, $r = x$

So if $|x| < 1$

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

Any series like this is called a power series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

(increasing powers of x)

The Taylor series of a function $f(x)$ centered at $x = 0$ is

$$f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

ex Taylor series of $f(x) = \frac{1}{1-x}$ centered at $x=0$

$f(x) = \frac{1}{1-x} = (1-x)^{-1}$ $\qquad f(0) = 1$

$f'(x) = +(x-1)^{-2}$ $\qquad f'(0) = 1 = 1!$

$f''(x) = +2(x-1)^{-3}$ $\qquad f''(0) = 2 = 2!$

$f'''(x) = +3\cdot 2(x-1)^{-4}$ $\qquad f'''(0) = 6 = 3!$

$f^{(4)}(x) = +4\cdot 3\cdot 2(x-1)^{-5}$ $\qquad f^{(4)}(0) = 4!$

$$1 + \frac{1}{1!}x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \frac{4!}{4!}x^4 + \dots$$

$$= 1 + x + x^2 + x^3 + x^4 + \dots$$

Three options

① A function and its Taylor Series agree everywhere. (ex: e^x, $\sin(x)$, $\cos(x)$)

② A function and its Taylor Series agree only in some interval
(ex. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ only when $-1 < x < 1$)

③ A function and its Taylor Series agree only at one point, (where its centered) ex

Taylor series of e^x centered @ $x=0$

$f(x) = e^x$ $\quad$ $f(0) = 1$

$f'(x) = e^x$ $\quad$ $f(0) = 1$

$\vdots$

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = e^x$$

Note: Partial sums of Taylor Series are Taylor Polynomials.

So Taylor Series can be used for approximation

Sometimes calculating higher derivatives can be troublesome.

We can use the Taylor series of a function to obtain Taylor series for related functions.

ex

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

Take T.S. for $\frac{1}{1-x}$ and substitute $(-x)$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{1+x} = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 - \cdots$$

ex Taylor series of

$$\frac{x}{1+x} = x\left(\frac{1}{1+x}\right)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

$$\frac{x}{1+x} = x(1 - x + x^2 - x^3 + x^4 - \dots)$$

$$= x - x^2 + x^3 - x^4 + x^5 - \dots$$

ex Taylor series of

$$\frac{1}{(1-x)^2}$$

Idea:

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x}$$

$$= \cancel{(1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)}$$

infinite foil.... no!

New idea:

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

so

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + \dots)$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

Try with e^x!

ex Find T.S. of $\ln(1-x)$

note

$$\int \frac{1}{1-x}\, d = -\ln(1-x) = (-1)(\ln(1-x))$$

integrate multiply by -1

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\int \frac{1}{1-x}\, dx = \int (1 + x + x^2 + x^3 + \cdots)\, dx$$

$$-\ln(1-x) + C = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

what is C?

plug in $x=0$ to both sides

$$-\ln(1) + C = 0 + \frac{0^2}{2} + \frac{0^3}{3} + \cdots$$

$$C = 0 \checkmark$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

ex Find the T.S. for $\frac{1}{(1-x)^3}$

$f(x) = \frac{1}{1-x}$

take 2 derivatives and divide by 2

$f'(x) = \frac{1}{(1-x)^2}$

$1 + x + x^2 + x^3 + x^4 + \cdots$

$f''(x) = \frac{2}{(1-x)^3}$

$\frac{d}{dx}$ $1 + 2x + 3x^2 + 4x^3 + \cdots$

d/dx $2 + 3\cdot 2x + 4\cdot 3x^2 + \cdots$

$\times 1/2$ $1 + \frac{3\cdot 2}{2} x + \frac{4\cdot 3}{2} x^2 + \frac{5\cdot 4}{2} x^3 + \frac{6\cdot 5}{2} x^4 + \cdots$

$+ \frac{(n+2)(n+1)}{2} x^n + \cdots$