# 11.2 Newton's Method / Algorithm

Important ~~idea~~ question i~~n~~ solving equations comes down to finding zeros or roots

For a function $f(x)$

if $f(r) = 0$ we call $r$ a root/zero

ex/

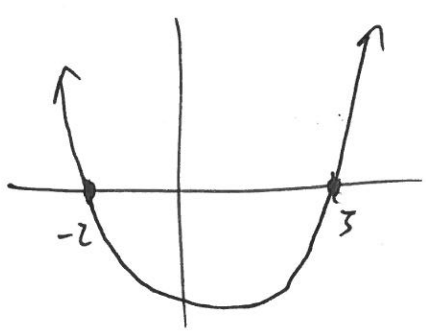
$$f(x) = x^2 - x - 6$$

$$= (x-3)(x+2)$$

so $f(3) = 0$ ; $f(-2) = 0$ — 3 & -2 are roots.

Graphically this is where the graph crosses the x-axis

ex/




$f(x) = x^2 - x - 6$

Sometimes we can factor or use information about the function

ex/ $f(x) = e^{x} - 1$ $\qquad e^{0} = 1$

so

$$f(0) = 0$$

and 0 is a root

ex/ $f(x) = \sin(x)$ has roots

at $x = 0, \pi, 2\pi, \ldots$

but real life can get complicated...

ex/

$$f(x) = e^{x^3 - x} + \ln(x)$$

roots ????

We will need to approximate roots.

Use the first Taylor polynomial to approximate $f(x)$ if we know a root is "close" to $x_0$

$$p(x) = f(x_0) + f'(x_0)(x - x_0)$$

and find root of the first Taylor polynomial.

$$0 = f(x_0) + f'(x_0)(x - x_0)$$

$$0 = f(x_0) + f'(x_0)x - f'(x_0)x_0$$

$$f'(x_0)x = f'(x_0)x_0 - f(x_0)$$

$$\boxed{x = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

If $x_0$ is close to a root of $f(x)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

is usually a better approximation.

We can iterate this method to get better and better approximations!

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

then $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots$$

$$\boxed{x_{new} = x_{old} - \frac{f(x_{old})}{f'(x_{old})}}$$

ex $f(x) = x^3 - x - 2$ has a zero between 1 & 2

let $x_0 = 1$ and find the next 3 approximations

$f'(x) = 3x^2 - 1$

so

$$x_1 = x_0 - \frac{x_0^3 - x_0 - 2}{3x_0^2 - 1}$$

$$x_1 = 1 - \frac{1^3 - 1 - 2}{3(1)^2 - 1} = 1 - \frac{-2}{2} = 2$$

$$x_2 = 2 - \frac{2^3 - 2 - 2}{3(2)^2 - 1} = 2 - \frac{4}{11} = \frac{18}{11}$$

$$x_3 = \frac{18}{11} - \frac{\left(\frac{18}{11}\right)^3 - \frac{18}{11} - 2}{3\left(\frac{18}{11}\right)^2 - 1} \approx 1.530$$

(actual root is 1.521 to 3 decimal places) ex Use Newton's Method to approximate the decimal value of $\sqrt{2}$ (4 repititions)

note that $f(x) = x^2 - 2$ has a root at $\sqrt{2}$

Lets start with $x_0 = 1$

- $f'(x) = 2x$

$$x_1 = x_0 - \frac{x_0^2 - 2}{2x_0}$$

$$x_1 = 1 - \frac{1^2 - 2}{2(1)} = 1 - \left(-\tfrac{1}{2}\right) = 1.5$$

$$x_2 = 1.5 - \frac{1.5^2 - 2}{2(1.5)} \approx 1.4167$$

$$x_3 = 1.4167 - \frac{(1.4167)^2 - 2}{2(1.4167)} \approx 1.41422$$

$$x_4 = 1.41422 - \frac{(1.41422)^2 - 2}{2(1.41422)} \approx 1.41421$$

correct approximation up to 5 decimal places ex

Approximate the positive solution of

$$e^x - 4 = x$$

The $x$ where the above is satisfied will be a root of

$$f(x) = e^x - 4 - x$$

- Graph $f(x)$ - note a root ~~close to~~ in between 1 and 2, closer to 2

$f'(x) = e^x - 1$ and $x_0 = 2$

$$x_1 = x_0 - \frac{e^{x_0} - 4 - x_0}{e^{x_0} - 1}$$

$$x_1 = 2 - \frac{e^2 - 4 - 2}{e^2 - 1} \approx 1.78$$

$$x_2 = 1.78 - \frac{e^{1.78} - 4 - 1.78}{e^{1.78} - 1} \approx 1.75$$

$$x_3 = 1.75 - \frac{e^{1.75} - 4 - 1.75}{e^{1.75} - 1} \approx 1.749$$

will give an approximate solution.