~~De~~ Desmos Demo for

$f(x) = \sin(x)$

Taylor polynomial is a good approximation of a function near the point $x = a$

ex

Find the n^{th} Taylor polynomial of $f(x) = e^x$ at $x = 0$

$$
\begin{aligned}
f(x) &= e^x & f(0) &= 1 \\
f'(x) &= e^x & f'(0) &= 1 \\
&\vdots & &\vdots \\
f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1
\end{aligned}
$$

$$P_n(x) = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

Recall: The n^{th} Taylor polynomial of a function $f(x)$ centered at $x=a$ is

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As seen in last class, Taylor polynomials give good approximations of the functions they come from

Why is this?

Use $a=0$ for simplicity:

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

- $P_n(0) = f(0)$ so values agree at $x=0$

- $P_n'(x) = f'(0) + \frac{2f''(0)}{2!}x + \frac{3f'''(0)}{3!}x^2 + \dots + \frac{nf^{(n)}(0)}{n!}x^{n-1}$

  $P_n'(0) = f'(0)$ so 1st derivatives agree

- $P_n''(x) = f''(0) + \frac{3\cdot 2\cdot f'''(0)}{3!}x + \dots + \frac{n\cdot(n-1)f^{(n)}(0)}{n!}x^{n-2}$

  $P_n''(0) = f''(0)$ so 2nd derivatives agree at $x=0$

$$P_n^{(n)}(x) = \frac{n \cdot (n-1)(n-2)\cdots(2)(1) f^{(n)}(0)}{n!}$$

$$= \frac{n!}{n!} f^{(n)}(0)$$

$$= f^{(n)}(0)$$

$P_n^{(n)}(0) = f^{(n)}(0)$ so n^{th} derivatives agree at $x = 0$

So the Taylor polynomial $P_n(x)$ and $f(x)$ ~~share~~ have their first n derivatives equal at $x = 0$

(can do the same for $x = a$)

ex/

Find 3rd Taylor Polynomial of

xe^{3x} at $x=0$

$f(x) = xe^{3x}$ $\qquad$ $f(0) = 0$

$f'(x) = e^{3x} + 3xe^{3x}$ $\qquad$ $f'(0) = 1$

$f''(x) = 3e^{3x} + 3e^{3x} + 9xe^{3x}$ $\qquad$ $f''(0) = 6$

$f'''(x) = 9e^{3x} + 9e^{3x} + 9e^{3x} + 27xe^{3x}$ $\qquad$ $f'''(0) = 27$

$$P_3(x) = 0 + \frac{1}{1!}(x-0) + \frac{6}{2!}(x-0)^2 + \frac{27}{3!}(x-0)^3$$

$$= x + 3x^2 + \frac{9}{2}x^3$$

ex/ ~~Using~~ Use The 4^{th} Taylor polynomial of e^x centered at $x=0$ to approximate $e^{0.1}$

$$p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{0.1} \approx p_4(0.1)$$

$$e^{0.1} \approx 1 + 0.1 + \frac{0.1^2}{2!} + \frac{0.1^3}{3!} + \frac{0.1^4}{4!}$$

$$\cancel{1 + .01 + \tfrac{1}{2}(0.01) + \tfrac{1}{6}(0.001)}$$

$$1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6} + \frac{0.0001}{24}$$

$$p_4(0.1) = 1.105170\boxed{8}\ldots$$

$$e^{0.1} = 1.105170\boxed{9}\ldots$$

agree up to the 6^{th} decimal place!

ex/ Find the 2^{nd} T.P. of

$f(x) = \ln(1+x^2)$ at $x=0$ and

Use it to approximate area under f

between $x=0$ and $x=\frac{1}{2}$

$f(x) = \ln(1+x^2)$ $\qquad f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{1+x^2} \cdot 2x$ $\qquad f'(0) = 0$

$f''(x) = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2}$ $\qquad f''(0) = \frac{2-0}{1} = 2$

$$P_2(X) = 0 + 0 \cdot x + \frac{2}{2!}x^2$$

$$\ln(1+x^2) \approx P_2(x) = x^2$$

So we can use $\int_0^{\frac{1}{2}} x^2\,dx$ to

approximate $\int_0^{\frac{1}{2}} \ln(1+x^2)\,dx$

$$\int_0^{\frac{1}{2}} \ln(1+x^2)\,dx \approx \int_0^{\frac{1}{2}} x^2\,dx = \left.\frac{x^3}{3}\right]_0^{\frac{1}{2}} = \frac{\frac{1}{8}}{3} = \frac{1}{24}$$

ex/ Find 4th Taylor Polynomial of $\ln(2-x)$ at $x=1$ and use it to estimate $\ln(.8)$

$f(x) = \ln(2-x)$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{2-x}(-1) = -(2-x)^{-1}$ $\qquad f'(1) = -1$

$f''(x) = (2-x)^{-2}(-1)$ $\qquad f''(1) = -1$

$f'''(x) = (-2)\cancel{(-1)}(2-x)^{-3}\cancel{(-1)}$ $\qquad f'''(1) = -2$

$f^{(4)}(x) = (-2)(-3)(2-x)^{-4}(-1)$ $\qquad f^{(4)}(1) = -6$

$$P_4(x) = 0 + \frac{-1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{-2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4$$

$\ln(.8) = \ln(2-1.2) = f(1.2)$

so $x = 1.2$

$$\ln(.8) \approx P_4(1.2) = -1(1.2-1) - \frac{1}{2}(1.2-1)^2 - \frac{2}{3!}(1.2-1)^3 - \frac{6}{4!}(1.2-1)^4$$

$$= -.2 - \frac{(.2)^2}{2} - \frac{2}{6}(.2)^3 - \frac{6}{24}(.2)^4$$

ex Find the 4^{th} Taylor Polynomial of

$f(x) = 3x^2 - 4x + 8$ at $x = -1$

$f(x) = 3x^2 - 4x + 8$ $\quad$ $f(-1) = 3 + 4 + 8 = 15$

$f'(x) = 6x - 4$ $\quad$ $f'(-1) = -10$

$f''(x) = 6$ $\quad$ $f''(-1) = 6$

$f'''(x) = 0$ $\quad$ $f'''(-1) = 0$

$f^{(4)}(x) = 0$ $\quad$ $f^{(4)}(-1) = 0$

$$P_4(x) = 15 + \frac{-10}{1!}(x+1) + \frac{6}{2!}(x+1)^2$$

Clean it up:

$$P_4(x) = 15 - 10x - 10 + 3(x^2 + 2x + 1)$$

$$= 5 - 10x + 3x^2 + 6x + 3$$

$$= \boxed{3x^2 - 4x + 8} = f(x) \text{ exactly.}$$