# Section 3.2

Using Matrices to solve systems of equations.
(any number of equations, any number of unknowns.)

Recall: A linear equation in 2 variables can be written
$ax + by = c$ → is a line

Lets generalize to linear equations in more than 2 unknowns

ex/ $3x + 4y - 5z = 2$

Sometime instead of $x, y, z$
we might write $x_1, x_2, x_3, x_4, \dots, x_n$

Recall: We can we write systems of equations as matrix multiplication equations

ex/
$$\begin{cases} 4x + 2y - 8z = 9 \\ -x - y - z = -1 \\ 4x + y - z = 5 \end{cases} \iff \begin{bmatrix} 4 & 2 & -8 \\ -1 & -1 & -1 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ 5 \end{bmatrix}$$

these are two different ways of representing the same system.

In 4.3 we talked about solving systems of eq using matrix inverses & matrix multiplication.

$$\begin{bmatrix} 4 & 2 & -8 \\ -1 & -1 & -1 \\ 4 & 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 4 & 2 & -8 \\ -1 & -1 & -1 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 2 & -8 \\ -1 & -1 & -1 \\ 4 & 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 9 \\ -1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & 2 & -8 \\ -1 & -1 & -1 \\ 4 & 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 9 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x &= 1 \\ y &= 1/2 \\ z &= -1/2 \end{aligned}$$

Great! We can use matrix inverses to solve systems.

What are some limitations:

- What if our matrix is singular?
  - singular matrices don't have an inverse.
- What if it's not a square matrix?

So this method only works when we have the same # of unknowns and equations, and if that matrix even has an inverse.

<u>New method:</u> Use the Augmented matrix and Elementary Row operations

What is the Augmented matrix?

ex/ $\begin{cases} 2x + 3y = 4 \\ -x + 3y = 2 \end{cases} \Longrightarrow \begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\Downarrow$ to write as Augmented matrix

$$\left[\begin{array}{cc:c} 2 & 3 & 4 \\ -1 & 3 & 2 \end{array}\right]$$

We will use Row operations to simplify this matrix:

<u>Row operations</u> are "things" "operations" we can do to the augmented matrix that does not change the system it represents.

<u>3 Types</u>

~~1~~ <u>Type 1</u> - we can replace a row by a multiple of itself $\quad R_i \to aR_i \quad a \neq 0$

ex/ $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 3 & 2 \end{bmatrix} \Longleftrightarrow \begin{aligned} 2x + 3y &= 4 \\ -x + 3y &= 2 \end{aligned}$

Replace $R_1$ with $2R_1$

$\begin{bmatrix} 4 & 6 & 8 \\ -1 & 3 & 2 \end{bmatrix} \Longleftrightarrow \begin{aligned} 4x + 6y &= 8 \\ -x + 3y &= 2 \end{aligned}$

(multiply a row by a number)
non-zero

Type 2: add a multiple of a row to a different row

$R_i \longrightarrow R_i \pm a R_j$

ex
$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 3 & 2 \end{bmatrix} \qquad \begin{cases} 2x + 3y = 4 \\ -x + 3y = 2 \end{cases}$$

~~add 2 to the first row~~

add the second row ×2 to the first row

$R_1 \rightarrow R_1 + 2R_2$

$$\begin{bmatrix} 2+(-2) & 3+(6) & 4+(4) \\ -1 & 3 & 2 \end{bmatrix} \qquad \begin{cases} 2x - (2x) + 3y + (6y) = 4 + (4) \\ -x + 3y = 2 \end{cases}$$

$$\begin{bmatrix} 0 & 9 & 8 \\ -1 & 3 & 2 \end{bmatrix} \qquad \begin{cases} 9y = 8 \\ -x + 3y = 2 \end{cases}$$

Type 3: we can switch the order of rows

ex
$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 3 & 2 \end{bmatrix} \leftrightarrow \begin{cases} 2x + 3y = 4 \\ -x + 3y = 2 \end{cases}$$

$\downarrow$

$$\begin{bmatrix} -1 & 3 & 2 \\ 2 & 3 & 4 \end{bmatrix} \leftrightarrow \begin{cases} -x + 3y = 2 \\ 2x + 3y = 4 \end{cases}$$

Lets use these to solve a system of equations

ex
$$\begin{cases} -\frac{2x}{3} + \frac{y}{2} = -3 \\ x/4 - y = \frac{11}{4} \end{cases} \Leftrightarrow \begin{bmatrix} -2/3 & \frac{1}{2} & -3 \\ 1/4 & -1 & 11/4 \end{bmatrix}$$

Multiply the first row by 6
and the second row by 4

$$\begin{bmatrix} \boxed{-4} & 3 & -18 \\ 1 & -4 & 11 \end{bmatrix} \iff \begin{cases} -4x + 3y = -18 \\ x - 4y = 11 \end{cases}$$

0

Idea is to use the first non-zero # in the first row and clear out everything else in its column.

Use $-4$ to clear out the 1

Use the 2nd type of row operation

$$R_2 \to R_2 + \tfrac{1}{4} R_1$$

→ we can also ~~multiply the secon~~

$$R_2 \to 4R_2 \to 4R_2 + R_1$$

But lets switch the rows first

$$\begin{bmatrix} 1 & -4 & 11 \\ -4 & 3 & -18 \end{bmatrix} \iff \begin{cases} x - 4y = 11 \\ -4x + 3y = -18 \end{cases}$$

$$R_2 \to R_2 + 4R_1$$

$$\begin{bmatrix} 1 & -4 & 11 \\ 0 & -13 & 26 \end{bmatrix} \iff \begin{cases} x - 4y = 11 \\ -13y = 26 \end{cases}$$

Lets see how much we can simplify this

Lets take $R_2 \to \frac{-1}{13} R_2$

$$\begin{bmatrix} 1 & -4 & 11 \\ 0 & \boxed{1} & -2 \end{bmatrix} \iff \begin{cases} x - 4y = 11 \\ y = -2 \end{cases}$$

Now lets use the 1 to clear out the $-4$ above it

$$R_1 \to R_1 + 4R_2$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix} \iff \begin{cases} x = 3 \\ y = -2 \end{cases}$$

Why is this better?

~~The~~ ~~real~~ One reason is ~~its~~ it helps us organize the system

But the real reason is that we can write this as an algorithm that a computer can use to solve systems quickly

- This process has a couple names
  - Gaussian Elimination
  - Row reduction
  - Gauss-Jordan reduction

From now on will be using technology to do this row reduction for us.