9-11-18 Day 6

# Review Matrix Addition & Scalar Multiplication

**Addition:** add up corresponding entries

ex/

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 8 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 3 \end{bmatrix}$$

subtraction works the same way

**Scalar multiplication:** multiply each entry of the matrix by ~~that~~ the scalar

ex/

$$2 \cdot \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

**Transposition:** swaps rows with columns and vice versa

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

recall we write entries in the $i^{th}$ row and $j^{th}$ column of $A$ as $a_{ij}$

so in our transpos $a_{ij} \rightarrow a_{ji}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad \cancel{A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{32} \end{bmatrix}} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

$3 \times 2$ $\qquad$ $2 \times 3$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

# Matrix Multiplication

We can multiply two matrices if the first matrix's number of columns = the second matrix's # of rows.

$A$ is a $(2 \times 3)$ matrix

$B$ is a $(3 \times 4)$ matrix

$$\underset{(2\times 3)\ (3\times 4)}{AB}$$ ✓ ok to multiply

$$\underset{(3\times 4)\ (2\times 3)}{BA}$$ ✗ not ok to multiply

$AB = C$ matrix $C$ is a $2 \times 4$ matrix

The resulting matrix has the same # of rows as our first matrix (left matrix) and the same # of columns as the second (right) matrix.

$$\underset{2\times 3}{A} \cdot \underset{3\times 4}{B} = \underset{2\times 4}{C}$$

ex

$A$ is a $4 \times 5$ matrix

$B$ is a $100 \times 3$ matrix

$C$ is a $3 \times 4$ matrix

$D$ is a $5 \times 100$ matrix

what matrix multiplications are allowed?

$$\underset{4\times 5\ \ 5\times 100}{AD} \rightarrow 4 \times 100$$

$B\ C$

$100\times3 \quad 3\times4 \rightarrow 100\times4$

$C\,A\,D$

$3\times4 \quad 4\times5 \quad 5\times100$

$3\times5 \quad 5\times100$

$3\times100$

To multiply these matrices we know what the dimension will be. $AB = C$

To find ~~$c_{ij}$~~ in our new matrix $c_{ij}$

we will take the $i^{th}$ row of matrix $A$ multiplied by the $j^{th}$ column of matrix $B$

- multiplying rows and columns

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ax + by$$

ex/

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 8 \\ 9 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$3\times2 \qquad 2\times2$

what is the entry in the 3rd row and 2nd column of $AB$?

$$\begin{bmatrix} 2 & 3 \\ 4 & 8 \\ 9 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 9\cdot3 + 2\cdot4$$

$$= 27 + 8 = 35$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 8 \\ 9 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 35 \end{bmatrix} = \begin{bmatrix} 8 & 18 \\ 20 & 44 \\ 13 & 35 \end{bmatrix}$$

$3\times2 \qquad 2\times2 \qquad 3\times2$

$$AB = \begin{bmatrix} 8 & 18 \\ 20 & 44 \\ 13 & 35 \end{bmatrix} \qquad BA \text{ isn't defined}$$

unlike normal multiplication, matrix multiplication is <u>not</u> commutative

that $3 \cdot 5 = 5 \cdot 3$ but $AB \neq BA$

$ab = ba$

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when are $AB$ and $BA$ defined?

when $A$ & $B$ are <u>square</u> matrices of the same dimension.

A square matrix is of dimension $n \times n$

ex/ $\left.\begin{matrix} 3\times 3 \\ 4\times 4 \\ 2\times 2 \end{matrix}\right\}$ square $\qquad \left.\begin{matrix} 3\times 2 \\ 1\times 4 \\ 20\times 2 \end{matrix}\right\}$ not square matrices

ex/

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}$$

$2\times 2 \qquad 2\times 2$

$$\underset{2\times 2}{AB} = \begin{bmatrix} 1\cdot 3 + (-1)5 & 1\cdot 0 + (-1)(-1) \\ 0\cdot 3 + 2\cdot 5 & 0\cdot 0 + 2\cdot(-1) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix}$$

$$\nparallel$$

$$\underset{2\times 2}{BA} = \begin{bmatrix} 3\cdot 1 + 0\cdot 0 & 3\cdot(-1) + 0\cdot 2 \\ 5\cdot 1 + (-1)0 & 5(-1) + (-1)2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 5 & -7 \end{bmatrix}$$

quick note: we can use matrices to represent linear systems of equations

ex/

$$\begin{cases} 12x + 20y = 2 \\ -5x - 2y = 3 \end{cases}$$

2 equations & 2 unknowns

write this as a matrix:

$$\begin{bmatrix} 12 & 20 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

but we can use matrices to represent any nnumber of equations with any number of unknowns.

ex/

$$\begin{cases} x + 2y = 4 \\ 3x - y = 2 \\ -x + 10y = -2 \end{cases}$$

as a matrix multiplication we would write

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

3 equations
2 unknowns

ex/

$$\begin{cases} 3x + 2y + 4z - w = 1 \\ x - 2y + 3z - 4w = 5 \\ 2x + 8y - 4z + 0w = -6 \end{cases}$$

$$\rightarrow \begin{bmatrix} 3 & 2 & 4 & -1 \\ 1 & -2 & 3 & -4 \\ 2 & 8 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}$$

- Zero matrix $O$

ex the $2\times 2$ zero matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

the $3\times 1$ zero matrix is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- If we add the zero matrix to any matrix

$$\underset{m\times n}{A} + \underset{m\times n}{O} = A$$

- what if we multiply a matrix with the zero matrix?

$$\underset{(m\times n)}{A}\ \underset{(n\times p)}{O} = \underset{(m\times p)}{O} \qquad \underset{m\times n}{O}\ \underset{n\times p}{A} = \underset{m\times p}{O}$$

ex/ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- Identity Matrix

A matrix ~~so that~~
we will denote as $I$
and $AI = A = IA$

The identity matrix is a square matrix with 1's on diagonal and zeros everywhere else

$2\times 2$ identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$3\times 3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$1\times 1$ $[1]$

ex/ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

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Matrix ~~Division~~ Inverse

- Matrix $A$ has inverse $A^{-1}$
- what do inverses mean for real numbers

ex/ whats the inverse of 7?

$7^{-1} = \frac{1}{7}$ $\qquad 7 \cdot \left(\frac{1}{7}\right) = 1$

For matrices

$$AA^{-1} = I = A^{-1}A$$

Matrices and their inverses come in pairs,

For a $2\times 2$ matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\quad A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

what is the inverse of

$$A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} ? \qquad A^{-1} = \frac{1}{-1-1}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

check:

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

what's the inverse of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ?

$$\frac{1}{1\cdot 0 - 0\cdot 1} = \frac{1}{0}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ does not have an inverse.

Matrices without inverses are called <u>singular</u>

(Matrices without inverses do not have a matching partner $\Rightarrow$ singular)

| <u>Matrices</u> | <u>Real numbers</u> |
|---|---|
| $AX = C$ | $ax = c$ |
| $A^{-1}(\quad) \quad A^{-1}(\quad)$ | $\frac{ax}{a} = \frac{c}{a}$ |
| $A^{-1}AX = A^{-1}C$ | $\rightarrow x = \frac{c}{a}$ |
| $IX = A^{-1}C$ | |
| $X = A^{-1}C$ | |

note: it's important to multiply on the same side

$$(AX) = (C)A^{-1}$$
$$A^{-1} \quad\quad (\;)$$

$$AXA^{-1} = CA^{-1}$$

do not cancel b/c X is in the way

$$AX = C$$
$$A^{-1}(\;) \quad (\;)A^{-1}$$

$$A^{-1}AX = CA^{-1}$$

This is just wrong

we can use this idea to solve systems of equations.