Day 5

9/6

Matrices

Matrix Addition, Scalar multiplication

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

ex

$$A = \begin{bmatrix} 3 & 5 & 9 \\ 4 & 8 & 15 \end{bmatrix}$$

A is a 2×3 matrix

$$B = \begin{bmatrix} 4 & 8 \\ 15 & 16 \\ 23 & 42 \end{bmatrix}$$

B is a 3×2 matrix

To refer to ~~a~~ specific entry we will use the following notation

a_{ij} to refer to the element in the i^{th} row and j^{th} column of A

$$A = \begin{bmatrix} 3 & 5 & 9 \\ 4 & 8 & 15 \end{bmatrix} \qquad a_{21} = 4$$

$$A = \begin{bmatrix} 3 & 5 & 9 \\ 4 & 8 & 15 \end{bmatrix} \qquad a_{12} = 5$$

for a 3x3 matrix we could describe all its entries with this notation.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We usually use capital letters for naming matrices and lower case for entries of said matrix

Two matrices are equal if they have the same dimensions <u>and</u> their entries are all equivalent.

ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

A is 3×2 matrix

B is a 2×3 matrix

A and B are not equal

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3 common types of matrices!

- Row matrix / Row vector:

  $1 \times n$ matrix

  $$R = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \end{bmatrix}$$

  R is a 1×5 matrix

- Column matrix / Column vector

  $m \times 1$ matrix

  $$C = \begin{bmatrix} 8 \\ 13 \\ 21 \\ 34 \end{bmatrix}$$ 4×1 matrix

- Square matrix

  $m \times m$

  $$P = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$ 3×3 matrix

Matrix Addition

To add (or subtract) matrices we add (or subtract) corresponding entries.

* We can only add matrices of the same dimensions

ex

$$\begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 8 \\ 4 & 2 \\ -8 & 0 \end{bmatrix} = \begin{bmatrix} 3-2=1 & 7 \\ 6 & 2 \\ -7 & 1 \end{bmatrix}$$

3x2 3x2 3x2

✓

$$\begin{bmatrix} 2 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 6 \end{bmatrix}$$

Scalar Multiplication

We can also a matrix by a real number. To do this we multiply each entry by said number

ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 15 \end{bmatrix}$$

We refer to these individual numbers as <u>scalars</u> (because they scale the entire matrix). In the above example, the matrix A was scaled up by a factor of 3.

<u>ex</u> Suppose we constructed a matrix represented weekly sales in Canadien dollars of two different stores in Canada.

	Vancouver	Quebec
Xbox One	2400	300
PS4	1500	900
Switch	3600	3000

$$S = \begin{bmatrix} 2400 & 300 \\ 1500 & 900 \\ 3600 & 3000 \end{bmatrix}$$

If we wanted to convert these sales into their USD equivalent, we could multiply S by a scalar.

1 canadian dollar is equal to 0.76 US dollars to choose our scalar as 0.76

$$0.76 \cdot S = 0.76 \begin{bmatrix} 2400 & 300 \\ 1500 & 900 \\ 3600 & 3000 \end{bmatrix} = \begin{bmatrix} 1824 & 228 \\ 1140 & 684 \\ 2736 & 2280 \end{bmatrix}$$

This is easy to do in Excel or in google Sheets

<u>ex</u>

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} x & y & w \\ z & \cancel{t+1} & 3 \end{bmatrix}$$

$t + 1$

What is $A + 3C$?

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} + 3\begin{bmatrix} x & y & w \\ z & t+1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} + \begin{bmatrix} 3x & 3y & 3w \\ 3z & 3t+3 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3x & 3y-1 & 3w \\ 3+3z & 3t+8 & 6 \end{bmatrix}$$

<u>Transpositions</u>

If A is an $m \times n$ matrix then its <u>transpose</u> is an $n \times m$ matrix where we switch the columns and rows. We can denote the transpose of A as A^T

ex $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

here a_{ij} (entry in the i^{th} row and j^{th} column) has the new position a_{ji} in the transpose

$a_{21} = 2$ in A^T, in the 1^{st} row second col we have 2 ✓

Check yourselves:

Is $(A+B)^T = A^T + B^T$?

What is $(A^T)^T$?

$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $B = \begin{bmatrix} 9 \\ 11 \\ 13 \end{bmatrix}$ $A+B = \begin{bmatrix} 10 \\ 13 \\ 16 \end{bmatrix}$

$(A+B)^T = \begin{bmatrix} 10 & 13 & 16 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$B^T = \begin{bmatrix} 9 & 11 & 13 \end{bmatrix}$

$A^T + B^T = \begin{bmatrix} 10 & 13 & 16 \end{bmatrix}$

4.2 Matrix Multiplication

First we will start by multiplying a row vector with a column vector

$$\begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \end{bmatrix}$$

$1 \times 2 \qquad 2 \times 1 \qquad 1 \times 1$

ex

Say you bought 3 Fall Out Boy albums on iTunes for \$13 each and 7 Fall Out Boy singles for \$2 each.

Who still purchases music through iTunes?

How much total did we just spend?

normally/ we can think

$$3 \cdot 13 + 7 \cdot 2 = \$53$$

we can symbolize via matrix multiplication

$$C = \begin{bmatrix} 13 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$C \cdot Q = \begin{bmatrix} 13 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 13 \cdot 3 + 2 \cdot 7 = \$53$$

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{array}{r} 13 \times 3 = 39 \\ 2 \times 7 = 14 \\ \hline 53 \end{array}$$

$$\begin{bmatrix} 13 & 2 \end{bmatrix}$$

To find the value of the ij^{th} entry in a product, we take i^{th} row of the first matrix multiplied by the j^{th} column of the second matrix.

ex

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 8 \\ 9 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad (\underbrace{3\times 2 \quad 2\times 2}_{\text{dimensions of AB}})$$

AB defined? AB is 3×2 matrix

$$AB = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \qquad m_{11} = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = [2\cdot 1 + 3\cdot 2] = 8$$

$$AB = \begin{bmatrix} 8 & 18 \\ 20 & 44 \\ 13 & 35 \end{bmatrix} \qquad m_{32} = \begin{bmatrix} 9 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \underset{27 \;+\; 8}{9\cdot 3 + 2\cdot 4} = 35$$

3rd row of A · 2nd col of B

$$m_{22} = \begin{bmatrix} 4 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \underset{12 \;+\; 32}{4\cdot 3 + 8\cdot 4} = 44$$

* when multiplying two numbers a, b $\quad a\cdot b = b\cdot a$ ✓

AB we found, BA here isn't even defined

* for matrices $A\cdot B \neq B\cdot A$

We could write linear equations in this way too

ex Consider the equation

$$6x + 9y = 10$$

we could express this as

$$\begin{bmatrix} 6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix}$$

Can we multiply

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad ?$$

No! This is not defined by matrix multiplication.

We need the # of columns of our first matrix to be equal to the # of rows of our second matrix.

ex we <u>can</u> multiply a 3x4 matrix with... 4x6 matrix ✓

4x9 matrix ✓

we <u>cannot</u> multiply a 3x4 matrix with... 3x4 matrix X

6x4 matrix X

If we look at the dimensions of two matrices $\underset{(m \times n)}{A}$ and $\underset{(r \times s)}{B}$, then

$A \cdot B$ is only defined

if $n = r$, ie these <u>inside</u> numbers agree.

in general: a $m \times n$ matrix can be multiplied by an $n \times p$ matrix

$$(\,m \times \underbrace{n \quad n}_{} \times p\,)$$

if A is 4×10

B is 10×15

AB is defined ✓, AB is 4×15 matrix

if A is 3×123

B is 123×4

AB is 3×4

$$(\,3 \times \underbrace{123 \quad 123}_{\checkmark} \times 4\,)$$

dimensions of AB