Research Statement

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1 Overview

My research interests are in applied geometry. Specific applications of geometry that interest me are in data and shape analysis. Questions that we may want to ask about these objects include whether two such objects are equivalent up to a certain group action, what kind of symmetries does the object have, and what kinds of other features can be detected?

One method used to detect equivalence under a given group action is the **differential signature** introduced by Calabi, Olver, Shakiban, Tannenbaum, and Haker [2]. The differential signature of an immersed submanifold is a set parameterized by a suitable number of **differential invariants**. These differential invariants are real-valued functions that, given a local parameterization of a submanifold, depend locally on the submanifold and derivatives with respect to its parameterization, that are invariant with respect to group actions on the ambient space.

By construction, the differential signature does not depend on the choice of parameterization of the original manifold and is invariant under the given group action, therefore, if two submanifolds are equivalent up to a group action, then the differential signatures of these two submanifolds will coincide. In contrast, the formulas for differential invariants on a submanifold depend on the choice of parameterization and therefore can be difficult to compare.

For example, the differential signature of a planar curve immersed in $\mathbb{R}^2$ under actions of the special Euclidean group of rotations and translations is parameterized by the signed Euclidean curvature κ and its derivative with respect to the Euclidean arclength κ_s.

Applications of this equivalence problem can be found in computer vision, where it is important for a computer to be able to detect whether two objects are the same. Common group actions to look at here include the special Euclidean, Euclidean, similarity, equi-affine, affine, and projective groups. Using equi-affine differential invariants, for example, has seen use in handwriting recognition [6]. Additionally, images taken by a pin hole camera with a fixed center are related by a projective transformation, and if the camera is sufficiently far away, an affine transformation relates different parallel projections from a fixed point [1].

The differential signature can also be used in detecting local symmetries. Points on the differential signature are assigned a **signature index** which counts the number of points on the original manifold that map to that point. For certain points on the signature, the signature index directly returns the number of local symmetries of the manifold. Olver in [11] examines the groupoid structure of local symmetries and their relationship to the differential signature.

Given a differential invariant, it is straightforward to evaluate on a submanifold. However the reverse is not as straightforward, and reconstructing a submanifold given an appropriate number of computed differential invariants often necessitates solving complicated differential equations. For the purposes of reconstructing families of manifolds which share a common differential signature, it is important to have a reliable way to approximate solutions to these differential equations.

Since some amount of noise is expected when applying the differential signature to object recognition, one important question is to investigate what "close" signatures tell about the "closeness" of the associated curves. Alternatively, joint invariants, functions of points in a space that is invariant under a group action on the space, can be used to build signature sets that do not require taking any derivatives. To do this, a suitable number of points on the manifold are chosen and the joint invariants on these points parameterize the signature set as those points vary [10]. These signatures therefore end up being much less susceptible to noise at the cost of a signature set with a higher dimension and greater computation time. Additionally, computation of a signature using differential invariants poses a challenge when given data is a point set that represents a sampling of a smooth manifold. In this case, joint differential invariants can be used to produce a group-invariant numerical scheme that approximates the differential signature [2].

2 Global Equivalence Problem for Planar Curves

A primary application of the differential signature is in determining if two planar curves are congruent up to some group action. While algebraic and analytic curves are uniquely identified by their differential signature up to a group action, other curves can cause problems even for the relatively simple special Euclidean group action consisting of only rotations and translations. In [9] Musso and Nicolodi demonstrate that in the special Euclidean case, for any closed differential signature, there is a 1-parameter family of non-congruent curves that share that differential signature. To construct this 1-parameter family they introduce curves which have intervals of constant curvature. As stated previously, since the special Euclidean differential signature is parameterized by the Euclidean curvature κ and its derivative with respect to arclength κ_s, the interval of constant curvature all maps to a single point on the signature.

My research has been primarily concerned with non-degenerate curves in $\mathbb{R}^2$. A non-degenerate curve is one such that its curvature $\kappa(s)$ has only isolated extrema. In my joint paper with Kogan [7] we formulate new conditions for a curve to be uniquely identified by its Euclidean signature, and introduce a method of generating and differentiating families of non-degenerate curves that share a common signature. Firstly, we prove that closed, non-degenerate curves with simple a differential signature is uniquely identified by its differential signature (see Figure 2). Additionally while a main advantage of the differential signature is that it is parameterization independent, it is also invariant under non-continuous reparameterizations, and so we demonstrate that non-congruent curves with identical non-simple signatures may end up tracing the same signature in different ways.

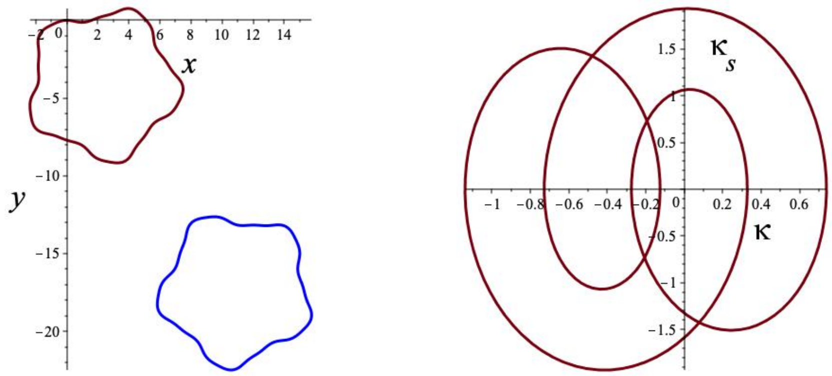

Figure 1: Two analytic curves that are congruent under $SE(2)$ and their shared Euclidean signature.

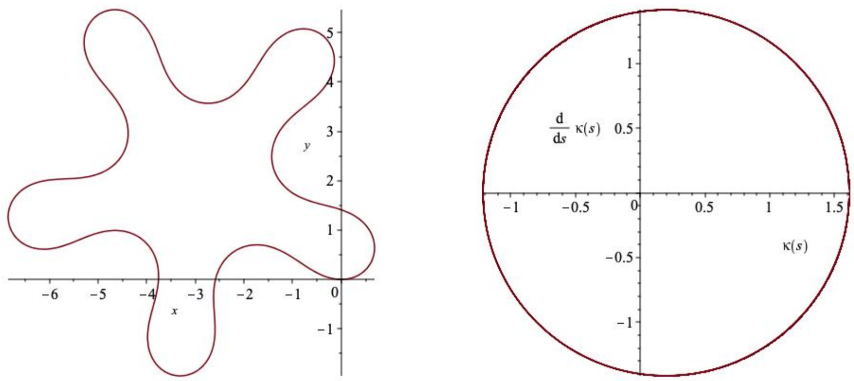

Figure 2: A non-degenerate curve with simple differential signature.

To differentiate the curves presented in Figure 3 we introduce the *signature quiver*, a directed graph attached to a signature where paths encode this needed extra information.

With this additional structure every curve has an associated path on their signature quiver which is enough to differentiate non-degenerate, non-congruent curves with the same Euclidean signature. For example, in Figure 3 the left curve has associated path $(adbc)^6$, while the right curve has associated path $(acacccbd)^3$. Additionally, introducing a notion of *weights* on edges of the signature quiver allows for identification of which paths correspond to closed curves, and as such allows for an enumeration of non-congruent, non-degenerate closed curves.

3 Global and Local Symmetry Detection

The global symmetries of a curve are those group elements that map the curve to itself. For the special Euclidean group, the global symmetries for closed curves are rotational symmetries. In Figure 3 it is not difficult to see that the right curve has 3 global symmetries, while the left curve has 6 global symmetries. These global symmetries are in fact baked into the words that describe paths on the signature quiver. We have shown that when the path of

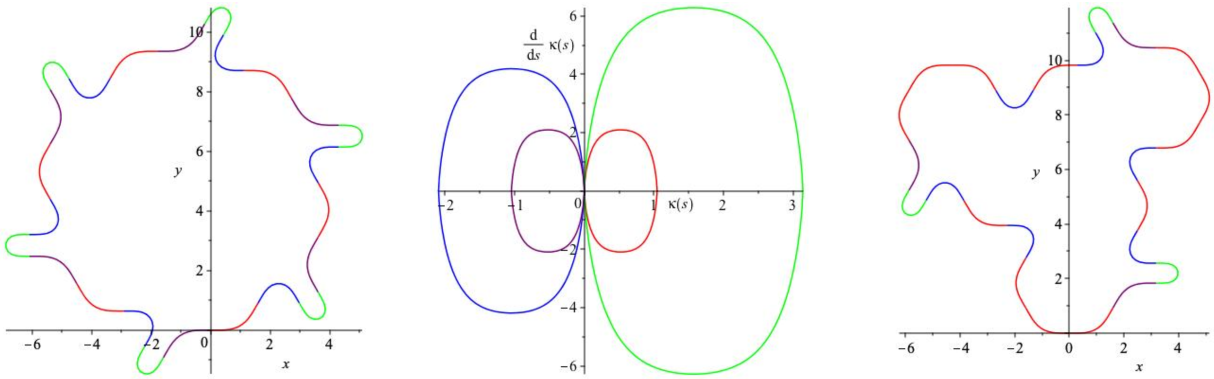

Figure 3: Two non-congruent curves with the same Euclidean signature.

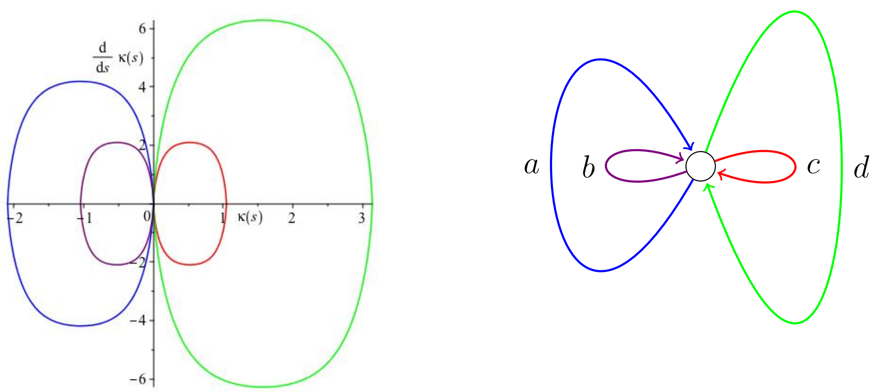

Figure 4: A signature and associated signature quiver.

a closed curve on the signature quiver, is written in its reduced form, the exponent gives exactly the number of global symmetries of the resulting curve. Furthermore, by assigning weights on edges of the signature quiver, we can determine which symmetry groups curves of a given signature can have.

Additionally, we show that the signature index of a curve, and thus its number of local symmetries at a point is also captured in the path the curve takes on the signature quiver. The number of local symmetries are encoded in the multiplicity of the edges. While the curves in Figure 3 have different global symmetries, away from points where $\kappa = 0$, the number of local symmetries on each point of the left curve is 6, while for points on the right curve may have 3,6, or 12 local symmetries each.

4 Curve Reconstruction from Curvatures

An important related project involves efficiently reconstructing curves based on their differential invariants. For the above work and in the work of [9], much of the examples are created by reproducing a curve given their special Euclidean curvature, so having an efficient way of reproducing a curve given any of its group-invariant curvatures is important for continued study in this area. Additionally, being able to reconstruct these curves can help address the problem of when "close" signatures result in "close" curves. While in the special Euclidean case this is a straightforward calculation, for other group-invariant curvatures curve reconstruction involves solving complicated differential equations. For example, in the equi-affine case, we obtain a system of linear differential equations. If the equi-affine curvature is constant the system can be easily solved to reconstruct parabolas, ellipses, and hyperbolas depending on the sign of the curvatures. If the curvature is non-constant, then the exact solutions are often not-available.

While mentoring four undergraduate students during an REU in the Summer of 2020 we investigated the particular case of reconstructing curves using their equi-affine curvature. We used Picard Iterations to successfully establish a process for reconstructing curves given their equi-affine curvature up to an arbitrary degree of accuracy. During this REU two students, Alex Kokot and Ian Klein were able to use this idea to determine bounds on the Hausdorff distance between two curves given the distance between their group-invariant curvatures and study the same questions for the signature [8].

5 Future Research Directions

Future work naturally includes studying signature quivers of curves under more complicated group actions that are relevant to computer vision like the equi-affine, affine, and projective groups. Determining how much additional structure is needed on the signature quiver in order to uniquely identify non-degenerate curves up to a group action is necessary for these larger groups. For transformation groups that contain more that one connected component, like the full Euclidean group, extra structure will be needed on the signature quiver to solve the global equivalence problem, as in these cases, simple signatures do not uniquely identify simple closed curves up to a group action. Additionally, the equi-affine curvature is not defined on a curve's inflection points. So, in order to apply the signature quiver to the global equivalence problem for non-convex curves additional structure for these points will be necessary. For classifying degenerate curves at the special Euclidean group, it seems that extra information can be added to the signature quiver along points that correspond to vertices which would greatly expand the class of curves that can be uniquely identified up to group action.

Another interesting avenue is to develop and study an analog of the signature quiver for higher dimensional submanifolds. For example, the special Euclidean signature of a surface embedded in $\mathbb{R}^3$ ends up being surface embedded in $\mathbb{R}^6$. Likely an analog of the signature quiver in this case will involve simplicial sets.

Additionally these differential signatures have the potential for interesting applications in machine learning based object recognition. Applying differential signatures to the curves representing the boundary of objects appears to be a natural way to pre-process data when images of these objects are likely rotated, scaled, or viewed from different angles.

While differential signatures have their advantages, another related object is the integral signature. As opposed to differential signatures which rely on derivatives and are thus very susceptible to noise, *integral signatures* introduced by Feng, Kogan, and Krim, [5] are much more robust under noise, but as of now lack the theoretical results that the differential signature has. An important problem is to establish theoretical results for this object and to further investigate properties of the integral signature which could prove it to be a useful tool in object recognition. Additionally the *path signature*, originally studied by Chen [3] and constructed via iterated integrals, has found application in time-series data. Diehl and Reizenstein show in [4] that these iterated integrals can be used to compute different group-invariant features when this data is represented as a path embedded in some ambient space. Exploring the relationship between the path signature and integral signature has the potential to lead to interesting extensions of the path signature that could result in stronger invariants that could be applied to object recognition.

References

[1] J. Burdis, I. Kogan, and H. Hong. "Object-Image Correspondence for Algebraic Curves under Projections". In: *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications [electronic only]* 9 (Mar. 2013). DOI: `10.3842/SIGMA.2013.023`.

[2] E. Calabi et al. "Differential and Numerically Invariant Signature Curves Applied to Object Recognition". In: *International Journal of Computer Vision* 26.2 (Feb. 1998), pp. 107–135. ISSN: 1573-1405. DOI: `10.1023/A:1007992709392`.

[3] K.-T. Chen. "Iterated path integrals". In: *Bull. Amer. Math. Soc.* 83.5 (Sept. 1977), pp. 831–879.

[4] J. Diehl and J. Reizenstein. "Invariants of Multidimensional Time Series Based on Their Iterated-Integral Signature". In: *Acta Applicandae Mathematicae* 164.1 (2019), pp. 83–122. ISSN: 15729036. DOI: `10.1007/s10440-018-00227-z`. arXiv: `1801.06104`.

[5] S. Feng, I. Kogan, and H. Krim. "Classification of Curves in 2D and 3D via Affine Integral Signatures". In: *Acta Applicandae Mathematicae* 109 (2008), pp. 903–937.

[6] T. Flash and A. Handzel. "Affine differential geometry analysis of human arm movements". In: *Biological cybernetics* 96 (July 2007), pp. 577–601. DOI: `10.1007/s00422-007-0145-5`.

[7] E. Geiger and I. A. Kogan. *Non-congruent non-degenerate curves with identical signatures.* 2020. arXiv: `1912.09597 [math.DG]`.

[8] A. Kokot and I. Klein. *Metrics and Uniqueness Criteria on the Signatures of Closed Curves.* 2020. arXiv: `2009.13004 [math.DG]`.

[9] E. Musso and L. Nicolodi. "Invariant Signatures of Closed Planar Curves". In: *Journal of Mathematical Imaging and Vision* 35.1 (Sept. 2009), pp. 68–85. ISSN: 1573-7683. DOI: `10.1007/s10851-009-0155-0`.

[10] P. Olver. "Joint Invariant Signatures". In: *Foundations of Computational Mathematics* 1 (Sept. 2000). DOI: `10.1007/s10208001001`.

[11] P. J. Olver. "The symmetry groupoid and weighted signature of a geometric object". In: *J. Lie Theory* 26.1 (2016), pp. 235–267. ISSN: 0949-5932.